Emergence of Fokker-Planck Dynamics within a Closed Finite Spin System

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Thermalization in closed quantum systems?

(Non-eq.) Thermodynamics

- autonomous dynamics of a few macrovariables
- attractive fixed point, equilibrium
- often describable by Fokker-Planck equations

Quantum Mechanics

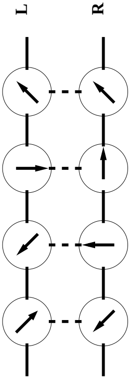
- autonomous dynamics of the wavefunction.
- no attractive fixed point (Schroedinger equation)
- Schroedinger is not Fokker-Planck

This puzzle (partially) triggered a lot of research:

"quantum typicality", eigenstate thermalization hypothesis (ETH), projection operator methods, open quantum systems, decoherence, Caldeira-Legett model, etc.

Quantum systems that show standard Fokker-Planck relaxation but are not of the "small system + large bath" type appear to be rare in the literature. Recent example: Ates et al., PRL **108**, (2012): magnetization in an Ising model with a transverse field decays according to Fokker-Planck but with a time-depedent FP-Operator.

spin-model



Heisenberg-type Hamiltonian: A ladder with anisotropic, *XXZ*-type couplings which are strong along the beams and weak along the rungs.

$$\hat{H} = \sum_{ij} J_{ij}(\hat{\sigma}_x^i \hat{\sigma}_x^j + \hat{\sigma}_y^i \hat{\sigma}_y^j + 0.6\, \hat{\sigma}_z^i \hat{\sigma}_z^j),$$

where $J_{ij} = 1$ for solid lines, $J_{ij} = \kappa = 0.2$ for dotted lines and $J_{ij} = 0$ otherwise. Total number of spins $N = 16$. The z-component of total magnetization S_z is conserved

We analyze: magnetization difference $\hat{x}$

$$\hat{x} = \frac{1}{2}\left(\sum_{l \in L} \hat{\sigma}_z^l - \sum_{r \in R} \hat{\sigma}_z^r\right)$$

eigenvalues of $\hat{x}$ within the subspace of vanishing total magnetization, $S_z = 0$: $X = -\frac{N}{4}, -\frac{N}{2} + 1, ... + \frac{N}{4}$.

Naive classical description

Assume there are rates at which mutual spin-flips, i.e., simultanous, contrariwise flips of adjacent spins occur. Let these rates be proportional to the square of the coupling constant between the adjacent spins.

Exploit local equilibrium due to time scale separation between leg-dynamics (fast) and rung-dynamics (slow) $\Rightarrow$

$$\text{Rates} \qquad R_{(X \to X \pm 1)} = \frac{\gamma \kappa^2 N}{2} \left(\frac{1}{2} \mp \frac{2X}{N} \right)^2$$

continuum limit, $N \to \infty, X \to \infty$, magnetization difference density $z := X/N$, Kramer-Moyal expansion:

$$\partial_t p(t,z) = -\partial_z((-\partial_z U(z)p) + \frac{1}{2}\partial_z^2(D(z)p) + \mathcal{O}(\partial_z^3)$$

$U(z) = \gamma\kappa^2 z^2$, $D(z) = \gamma\kappa^2(1/4 + 4z^2)/N$.

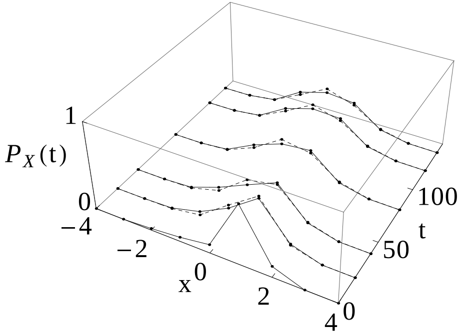
Almost like a Brownian particle in a parabolic potential.

Exact result vs. naive description

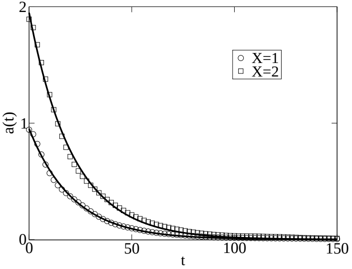
initial states

$$\hat{\rho}_X(0) = \frac{1}{Z}\hat{P}_{(0,2)}\hat{P}_X\hat{P}_{(0,2)}$$

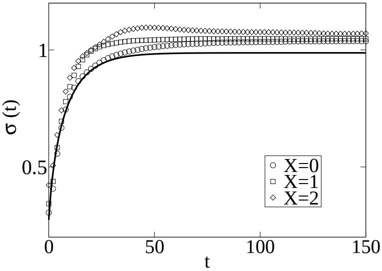
$\hat{P}_X$: projector onto subspace X
$\hat{P}_{(0,2)}$: projector onto energy intervall



mean of X



width of X



Do we understand those numerical findings?

We try to! This effort involves the TCL projection operator method

projection superoperator

$$\mathcal{P}\hat{\rho} = P_X \frac{\hat{P}_X}{d_X}, \quad P_X = \text{Tr}\{\hat{P}_X \hat{\rho}\}$$

$$d_X = \text{Tr}\{\hat{P}_X\} \quad \mathcal{P}^2 = \mathcal{P}$$

going through the formalism yields:

$$\dot{P}_Y = \sum_{X \neq Y} R^{TCL}_{Y,X}(t) P_X - \sum_{X \neq Y} R^{TCL}_{X,Y}(t) P_Y$$

realistically computable are 2. order rates:

$$R^{TCL2}_{Y,X}(t) := \int_0^t C_{Y,X}(t') dt'$$

time dependence: only generated by $\hat{H}_0$,
here: legs
$\hat{V}$: interaction, here: rungs

$$C_{Y,X}(t') = \frac{\kappa^2}{d_X} \text{Tr}\{[\hat{V}(t'), \hat{P}_Y][\hat{V}(0), \hat{P}_X]\}$$

Initial correlation functions are proportional to naive rates:

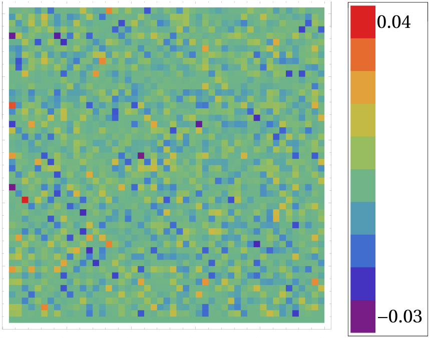
$$C_{Y,X}(0) = \delta_{Y,X\pm1} \frac{R_{X\to X\pm1}}{4\gamma}$$

This result is not restricted to this model.

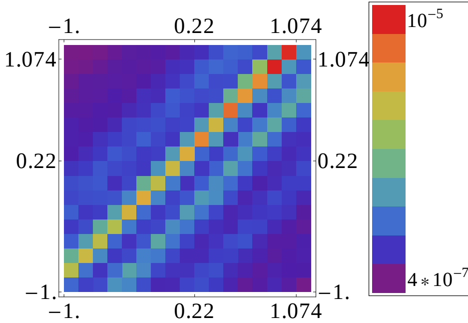
Do we understand those numerical findings?

There are more conditions on the validity of 2.order descriptions than just time-scale separation (Van Hove, Bartsch et al.): The interaction matrix must show features of a matrix the elements of which are drawn a random. This seems to hold here:

"random" fine structure of transition matrix



smooth coarse structure of transition matrix



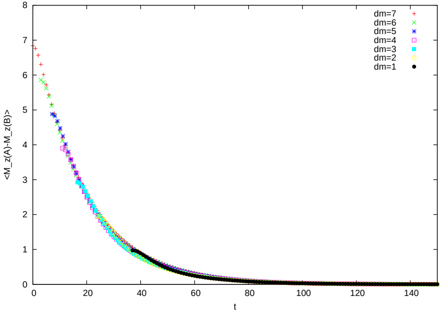
What about bigger systems?

- numerics: the quantum evolution of a pure state may be more easy to compute than the evolution of a mixed state
- dynamical typicality: adequate random pure states may "mimic" the dynamics of mixed states

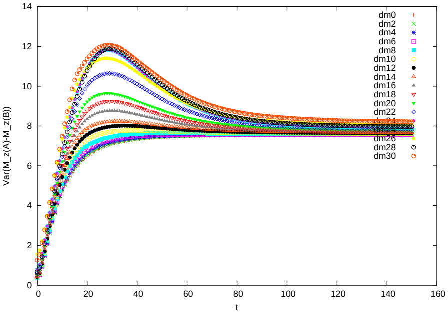
We use an iterative Chebyshev scheme to implement "Schroedinger-type" propagation

Initial states: $|\psi(0)\rangle \propto e^{-(\hat{H}-E)^2\tau}\hat{P}_X|\phi\rangle$, with $|\phi\rangle$ random, $E = 0$

shifted expectation values of X, $N = 30$



variances of X, $N = 30$



How is that comparable to two cups of coffee thermalizing each other?

- The largest inital $X(0)$ yielding "Markovian" decaying expectation values $\langle \hat{x}(0) \rangle$ appears to scale as $\propto N$.
- The maximun width δX during this decay appears to scale as $\propto \sqrt{N}$

Are the final width δX truely independent of the initial state and what does that imply for the ETH?

Since
$\langle \phi | \hat{x}^2(t) | \phi \rangle \langle \phi | \hat{x}^2 | \phi \rangle \approx \mathrm{Tr}\{\hat{x}^2(t)\hat{x}^2\}/d$
(typicality) and for large times
$\mathrm{Tr}\{\hat{x}^2(t)\hat{x}^2\} \to \sum_n |\langle n|\hat{x}^2|n\rangle|^2$
there are ways to infer the variance σ^2 of the distribution of $\langle n|\hat{x}^2|n\rangle$ within some energy regime from pure state evolutions.

more about this:
Europhys. Lett., 101, 10011 (2013)

Thank you for your attention!

relative spread of $\langle n|\hat{x}^2|n\rangle$, i.e., σ^2/N