The typicality approach to thermodynamical relaxation in quantum systems.

Jochen Gemmer

Universität Osnabrück

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Why typicality?
The traditional view on relaxation / 2nd law of thermodynamics:

\[ \hat{\rho} = |\psi\rangle\langle\psi| \quad \Rightarrow \quad \hat{\rho} = \frac{1}{Z} e^{-\frac{\hat{H}}{kT}}, \quad \hat{\rho} = \frac{1}{Z} \delta(\hat{H} - E) \]

CM: \[ \rho(x, p) = \delta(x-x_0)\delta(p-p_0) \quad \Rightarrow \quad \rho = \frac{1}{Z} e^{-\frac{H(x, p)}{kT}}, \quad \rho = \frac{1}{Z} \delta(H(x, p)-E) \]

problems:
CM: invariance of Von Neumann entropy, ergodicity, mixing, etc.
QM: (framework of open quantum systems) large (stationary, broad band) environment, adequate weak coupling, pertinent factorizing initial state, etc.

Typicality:
\[ \rho = |\psi\rangle\langle\psi| \quad \text{does not evolve into} \quad \hat{\rho} = \frac{1}{Z} e^{-\frac{\hat{H}}{kT}}, \quad \hat{\rho} = \frac{1}{Z} \delta(\hat{H} - E) \]

but \[ \langle\psi|\hat{A}(t)|\psi\rangle \approx \frac{1}{Z} \text{Tr}\{e^{-\frac{\hat{H}}{kT}} \hat{A}\} \]

for very many (all?) \( \hat{A} \)
The typicality scenario

\[ \bar{x} : \text{microstates in state space, AR: accessible region due to constants of motion, } \]
\[ f(\bar{x}) : \text{considered quantity} \]

- **Average:** \[ E_{AR}(f) = \int_{AR} f(\bar{x}) \, dV_{\bar{x}} \]
- **Variance:** \[ V_{AR}(f) = E_{AR}[f^2] - E_{AR}[f]^2 \]

**Typicality:** \[ V_{AR}^{\frac{1}{2}}[f] \leq f_{\text{max}} - f_{\text{min}} \]
⇒ relative frequency of stats featuring \( f(\bar{x}) \approx E_{AR}(f) \) is high

connection to dynamics possible if \[ \dot{\bar{x}} = \overline{G}(\bar{x}) \quad \text{div}_{\bar{x}} \overline{G} = 0 \]

invariance of state space volume, no ergodicity, mixing, etc.
Typicality in QM

basics and non-composite systems state:

$$|\psi\rangle = \sum_{n} \psi_n |n\rangle = \sum_{n} (\eta_n + i \xi_n |n\rangle)$$

$$\vec{x} = \{\eta_n, \xi_n\} : \eta_n, \xi_n : \text{real cartesian coordinates}$$

dynamics: Schrödinger equation, $$\dot{\vec{x}} = \overline{H} (\vec{x})$$, $$\text{div}_\vec{x} \overline{H} = 0$$

accessible region: \( \hat{\Pi}_\alpha \): projective constants of motion (invariant subspaces), e.g., spanned by energy eigenstates, spanned by states featuring equal particle number, etc.

$$\hat{\Pi}_\alpha^2 = \hat{\Pi}_\alpha$$, $$[\hat{H}, \hat{\Pi}_\alpha] = 0$$, $$N_\alpha = \text{Tr} \{ \hat{\Pi}_\alpha \}$$, $$\text{Tr} \{ \hat{\Pi}_\alpha, \hat{\Pi}_\beta \} = N_\alpha \delta_{\beta\alpha}$$

AR: \( \{ \langle \psi | \hat{\Pi}_\alpha | \psi \rangle = W_\alpha \} \) occupation probabilities of subspaces conserved

considered quantity: $$f(\vec{x}) = \langle \psi | \hat{A} | \psi \rangle$$
Hilbert space average of observables:

\[ E_{AR}[f] \equiv \frac{\langle \psi | \hat{A} | \psi \rangle}{\langle \psi | \psi \rangle} = \text{Tr}\{ \hat{A} \hat{\Omega} \} , \quad \hat{\Omega} = \sum_{\alpha} \frac{W_{\alpha}}{N_{\alpha}} \hat{N}_{\alpha} \]

Boltzmann state featuring “equal a priori probabilities

Hilbert space variance of observables:

\[ V_{AR}[f] \equiv \Delta_{H}^2(\langle \hat{A} \rangle) = \sum_{\alpha \beta} \frac{W_{\alpha} W_{\beta}}{N_{\alpha}(N_{\beta} + \delta_{\alpha \beta})} \left( \text{Tr}\{ \hat{A}_{\alpha \beta} \hat{A}_{\alpha \beta}^{\dagger} \} - \delta_{\alpha \beta} \frac{\text{Tr}\{ \hat{A}_{\alpha \alpha} \}^2}{N_{\alpha}} \right) \]

consider, e.g.:

\[ \langle \psi | \hat{N}_{\alpha} | \psi \rangle = 1 \quad \Rightarrow \quad \Delta_{H}^2(\langle \hat{A} \rangle) = \frac{1}{N_{\alpha} + 1} \Delta_{S}^2(\hat{A}) \]

typicality: requires high dimensional space, bounded spectra
typicality of states: (squared) distance of most states within the AR to some “typical state” is small. \( \hat{\Omega} \): candidate for a typical state

mean squared distance: \[ \Delta_H^2(\hat{\rho}) = \left[ \text{Tr}\{(\hat{\rho} - \hat{\Omega})^2\} \right]_{AR} \]

for non-composite systems

\[ \Delta_H^2(\hat{\rho}) = 1 - \text{Tr}\{\hat{\Omega}^2\} \approx 1 \quad \Rightarrow \]

for Boltzmann-type \( \hat{\Omega} \)’s not small at all.

Conclusion on non-composite systems:
Many observables may relax, but the state does not. There is no increase of Von Neumann entropy.

composite systems: \[ H = H_S + H_E + V \]

projective subspaces:
sys.: \( \hat{\Pi}_A : E_A - \frac{1}{2}\delta_A \leq E \leq E_A + \frac{1}{2}\delta_A \), env.: \( \hat{\Pi}_B : E_B - \frac{1}{2}\delta_B \leq E \leq E_B + \frac{1}{2}\delta_B \)

definition of global projective subspaces: \[ \hat{\Pi}_\alpha = \hat{\Pi}_A \hat{\Pi}_B \]
Microcanonical Scenario: \[ [H_S, H] = 0 \]

accessible region \( AR : \{ \langle \psi | \hat{N}_A | \psi \rangle = W_A , \langle \psi | \hat{N}_B | \psi \rangle = W_B \} \)

energies in system/environment separately conserved, no energy in coupling. Candidate for the typical local system state:

\[
\Omega \equiv \sum_A \frac{W_A}{N_A} \hat{N}_A \quad \text{local Boltzmann state}
\]

⇒

mean squared distance:
\[
\hat{\Delta}_H^2(\hat{\rho}) = \sum_B \frac{W_B^2}{N_B} \left( 1 - \sum_A \frac{W_A^2}{N_A^2} \right)
\]

scaling of upper bound with subsystem sizes:

\[
N_A \rightarrow \times N_A \quad N_B \rightarrow \times N_B \quad \Rightarrow \quad \Delta_H^{2+}(\hat{\rho}) \rightarrow \frac{1}{y} \Delta_H^{2+}(\hat{\rho})
\]

For large environments there is full typicality of state, increase of entropy, etc.
energy exchange scenario

total energy subspaces: \[ \hat{\Pi}_E = \sum_{E_A + E_B \approx E} \hat{\Pi}_A \hat{\Pi}_B \]

accessible region \[ AR : \{ \langle \psi | \hat{\Pi}_E | \psi \rangle = W_E \} \]

only total energy is conserved, no energy in the coupling

Candidate for the typical local system state:

\[ \Omega \equiv \text{Tr}_{env} \left\{ \sum_{E} \frac{W_E}{N_E} \hat{\Pi}_E \right\} \]

scaling of upper bound with subsystem sizes:

\[ N_A \rightarrow x N_A \quad N_B \rightarrow x N_B \quad \Rightarrow \quad \Delta^2_H(\hat{\rho}) \rightarrow \frac{1}{y} \Delta^2_H(\hat{\rho}) \]

For large environments there is full typicality of state, increase of entropy, etc.
Canonical Scenario:

What about the standard canonical Gibbs state?

A state density in the environment yielding $N_B \propto e^{cE}$ can be expected for environments made of weakly interacting subsystems.

In this case one finds for the typical energy exchange state:

$$\Rightarrow \Omega = \frac{1}{Z} e^{-cE_A \hat{N}_A} \approx \frac{1}{Z} e^{-\frac{H}{kT}}, \quad c \approx \frac{1}{kT}$$

Comment on relaxation in composite systems:

All relaxation in composite systems, regardless of the strength of the interaction, is due to increasing correlations/entanglement.
consider more general AR:

$$\text{AR : } \{ \langle \psi | \hat{A} | \psi \rangle = A, \quad \langle \psi | \psi \rangle = 1 \}$$

$\hat{A}$: any hermitian Observable.

How differently will the various $\langle \psi | \hat{A} | \psi \rangle$ from the AR evolve?

$$\Delta_H^2 (\langle \psi | \hat{A}(t) | \psi \rangle) = \left[ \langle \psi | \hat{A}(t) | \psi \rangle^2 \right]_{AR} - \left[ \langle \psi | \hat{A}(t) | \psi \rangle \right]_{AR}^2$$

Hard to answer. But consider:

$$|\phi\rangle = \left( \hat{1} + \frac{d}{(1 + d^2)} \hat{A} \right) |\theta\rangle, \quad \text{Tr}\{\hat{A}\} = 0, \quad \text{Tr}\{\hat{A}^j\} = c_j$$

with $c_2 = 1$ and $\frac{c_i}{N}$ approximately independent of $N$
accessible region for the $|\theta\rangle$'s:

$$\begin{align*}
AR: & \quad \{\langle \theta | \theta \rangle = 1 \} \Rightarrow \\
\mathbb{E}[\langle \phi | \phi \rangle]_{AR} &= 1, \\
\mathbb{E}[\langle \phi | \hat{A} | \phi \rangle]_{AR} &= 2d,
\end{align*}$$

$$\begin{align*}
\Delta^2_H(\langle \phi | \phi \rangle) &\propto \frac{1}{N} \\
\Delta^2_H(\langle \phi | \hat{A} | \phi \rangle) &\propto \frac{1}{d^2 N}
\end{align*}$$

$N$ large: almost all $|\phi\rangle$ are from the AR with $A = 2d$!

result for the dynamics of the $|\phi\rangle$'s:

$$\begin{align*}
\sqrt{\Delta^2_H(\langle \phi | \hat{A}(t) | \phi \rangle)} \frac{2d}{2d} &\leq \frac{1}{\sqrt{N}} \\
\mathbb{E}[\langle \phi | \hat{A}(t) | \phi \rangle]_{AR} &= \text{Tr}\{\hat{A} \hat{\rho}(t)\}, \quad \hat{\rho}(0) = \hat{1} + 2d \hat{A}
\end{align*}$$

For large $N$ almost all $\langle \psi | \hat{A}(t) | \psi \rangle$ evolve very similar!

The average evolution of all $\langle \psi | \hat{A}(t) | \psi \rangle$ may possibly be computed with projection methods $\Rightarrow$ The inhomogeneity in the NZ-equation may almost always be neglected.
“generic evolutions”:

\[ N = 1000 \]

“standard” weak coupling

Exponentieller Zerfall, Nges=1000

strong coupling

Nicht-Exponentieller Zerfall, Nges=1000

“pathological” weak coupling

Kein Zerfall, Nges=1000

The “take home message”:
According to the typicality approach relaxation is not a necessity, but something that is extrem likely to happen in complex systems.

more information, publications: ask me or visit our webpage.

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