Boltzmann approach to transport in weakly interacting, near integrable, 1-d fermionic systems

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The model and some of its transport properties

spinless fermions Hubbard-type tight-binding model plus next nearest neighbor hopping

$$H = J \sum_{n=1}^{N} \Big[-\frac{1}{2}(a_n^\dagger a_{n+1} + b \cdot a_n^\dagger a_{n+2} + \text{h.c.}) + \Delta \cdot a_n^\dagger a_n a_{n+1}^\dagger a_{n+1} \Big]$$

What is known (more or less for sure)?

- for $b = 0$ and any Δ the model is "integrable": Bethe ansatz applies, infinitely many local conserved quantities, no Wigner-Dyson level statistics, etc.
- linear response roughly states that transport is ballistic if $\sum_n \text{Tr}(\hat{J}\hat{Q}_n) \neq 0$ $\Rightarrow$ integrable systems tend do be ballistic ($\approx$ feature infinite conductivity, diffusion constants, etc.)

Even for $b = 0$ the transport properties may depend on interaction Δ, filling f and temperature T.

- for $\Delta > 1$ the system is "gapped" and non-ballistic, i.e., diffusive for all (most) fillings and temperatures
- for $\Delta < 1$ the system is most likely ballistic for all cases except the following:
- for $\Delta < 1$, half filling $f = 0.5$ and finite temperature $T \neq 0$ is currently under debate

How to set up a (linearized) Boltzmann equation ?

We intend to investigate transport properties in the regime $\Delta << 1$, $b < 1$ by mapping the system onto a Boltzmann equation and extracting transport properties from the latter

Diagonalization of hopping model via Fourier transformation:

$$H = \sum_k e(k) a_k^\dagger a_k + \frac{1}{2} \sum_{k,l,q} \frac{W(q)}{N} \cdot a_{k+q}^\dagger a_{l-q}^\dagger a_k a_l \ ,$$

$$e(k) = J(\cos(k) + b \cdot \cos(2k)) \ , \quad W(q) = -2J\Delta \cdot \mathrm{e}^{-\imath q}$$

We try a TCL projection scheme in which we keep the deviations from equilibrium of the momentum occupation numbers as relevant variables:

$$\Delta_\mathsf{j} := a_\mathsf{j}^\dagger a_\mathsf{j} - f_\mathsf{j} \qquad d_\mathsf{j}(t) := \mathrm{Tr}\{\Delta_\mathsf{j} \rho(t)\}$$

We choose the following projection

$$\mathcal{P}\rho(t) = \mathrm{Tr}\{\rho(t)\}\rho^{\mathsf{eq}} + \sum_\mathsf{j} \frac{\rho^{eq} \Delta_\mathsf{j}}{\mathrm{Tr}\{\rho^{eq} \Delta_\mathsf{j}^2\}} d_\mathsf{j}(t)$$

Performing a nasty leading order (Δ^2) calculation of the correponding equation of motion leaves us with....

How to set up a (linearized) Boltzmann equation ?

...something that may be interpreted as the linearized scattering operator of a Boltzmann equation:

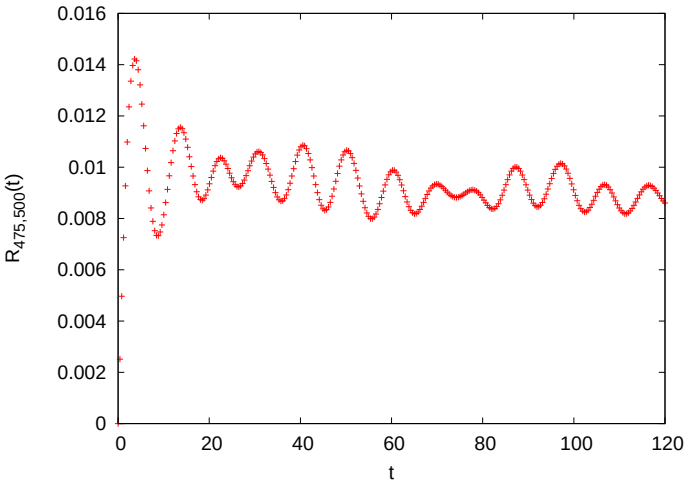
$$\dot{d}_i(t) = \sum_{k \neq i} R_{ik}(t) d_k(t) - \sum_{k \neq i} R_{ki}(t) d_i(t)$$

where the corresponding rates ($T = \infty$, $f = 0.5$) are given by

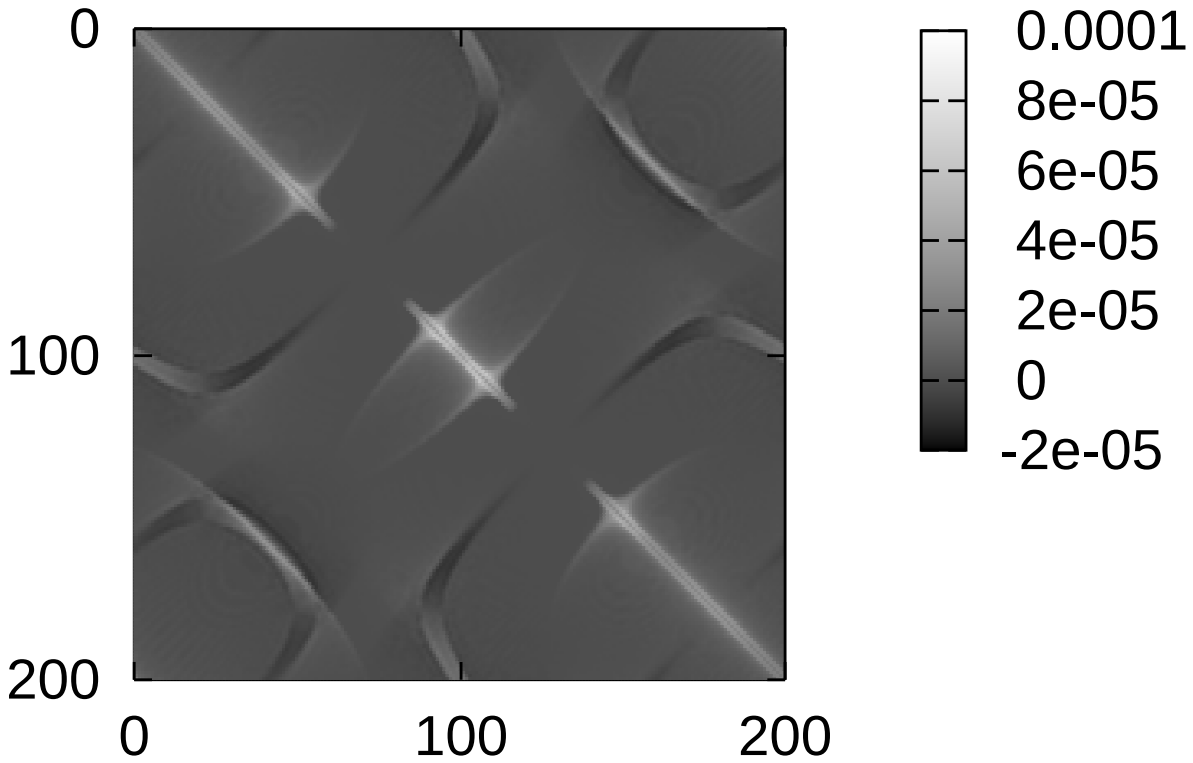
$$R_{\mathbf{k},\mathbf{i}}(t) = \frac{1}{2N^2\hbar^2} \sum_{\mathbf{l}} (\mathrm{Re}(W(\mathbf{i}-\mathbf{k})) - \mathrm{Re}(W(\mathbf{k}-\mathbf{l})))^2 t \mathrm{sinc}[\frac{1}{\hbar}(\varepsilon_{\mathbf{i}} + \varepsilon_{\mathbf{l}} - \varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{i}-\mathbf{k}+\mathbf{l}})t]$$
$$-\frac{1}{2}(\mathrm{Re}(W(\mathbf{l}-\mathbf{i})) - \mathrm{Re}(W(\mathbf{k}-\mathbf{l})))^2 t \mathrm{sinc}[\frac{1}{\hbar}(\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{i}} - \varepsilon_{\mathbf{i}+\mathbf{k}-\mathbf{l}} - \varepsilon_{\mathbf{l}})t]$$

In the limit of large times and large system sizes the rates should become time-independent (for non-pathological cases, i.e., no multiple zeroes in the sinc-argument)

Rates and scattering operators



above:
time evolution of one rate for
$b = 1, \Delta = 0.01, T = \infty, N = 600$
right:
scattering operator R
$b = 1, \Delta = 0.01, T = \infty, N = 200$

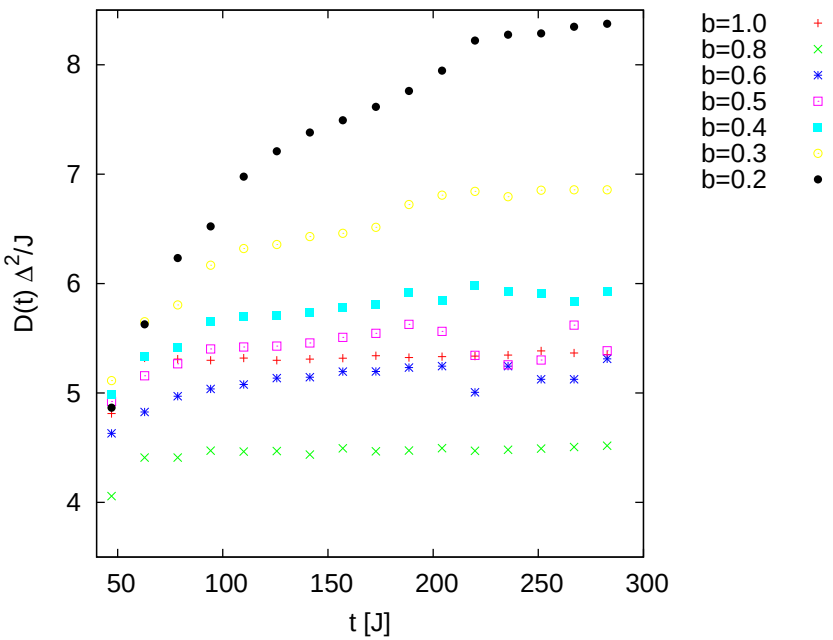


Diffusion coefficients

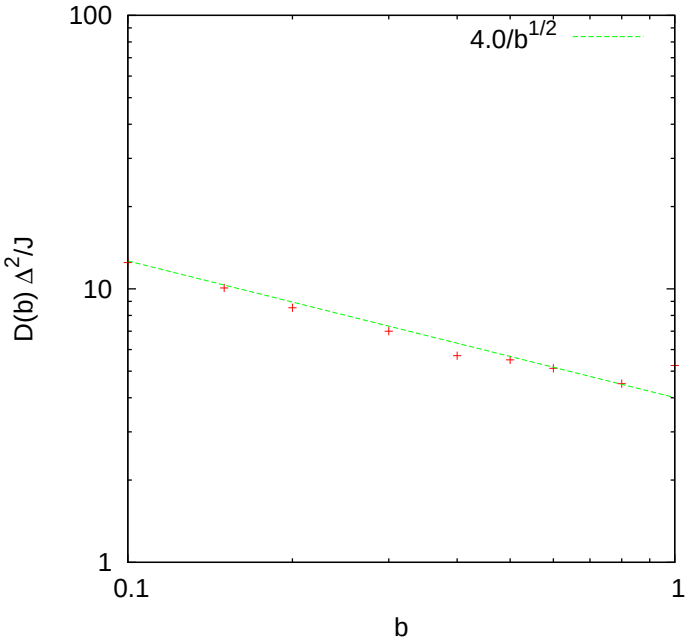
For regular linear Boltzmann equations (at $T = \infty$) the diffusion coefficient reads

$$D = -\frac{1}{N} v_{\mathbf{l}} R^{-1}_{\mathbf{lk}} v_{\mathbf{k}} \qquad \text{velocities} : v_{\mathbf{k}} = \frac{\partial \epsilon}{\partial \mathbf{k}}$$

"instantanous diffusion coefficients"



"long time diffusion coefficients"



It appears that with $b \longrightarrow 0$:

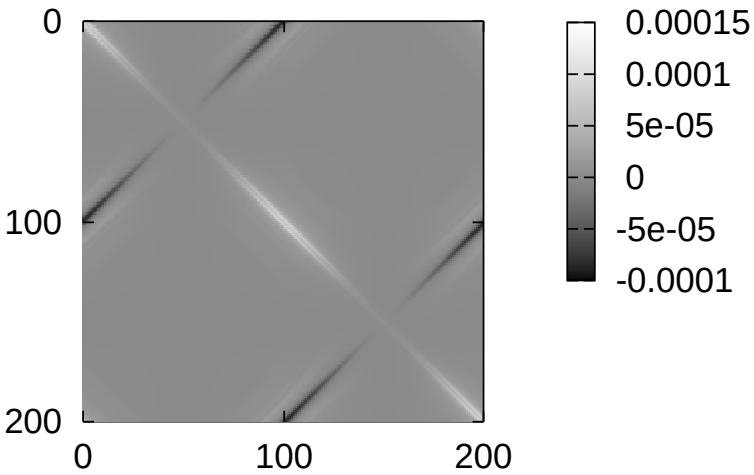
- the timescale on which the scattering operator becomes constant diverges
- the diffusion coefficient diverges as $D \propto b^{-1/2}$

Anisotropic chain without next nearest neighbor hopping

Taking a closer look at $b \longrightarrow 0$ we find that

- the argument in the sinc function in the rate-expression indeed appraoches multiple zeroes
- especially for $b = 0, N \longrightarrow \infty, t \longrightarrow \infty, t \longrightarrow \infty$ the scattering operator can be found analytically

$$R(j,k) \propto \frac{(\cos(k-j)+\cos(k+j))^2}{|\sin(k)-\sin(i)|}$$



"finite time sketch"

At $t \longrightarrow \infty$

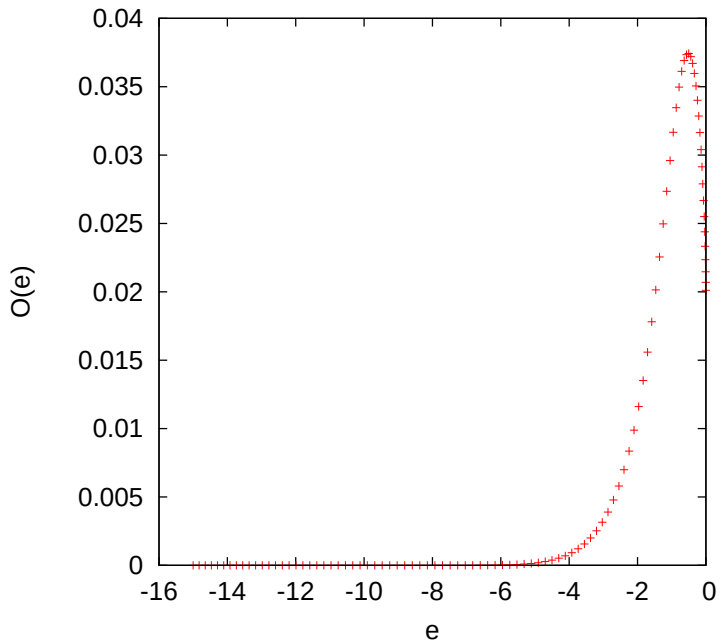
- the scattering operator diverges at some transitions: $j + k = \pi/2, j + k = -\pi/2$
- the scattering operator has at least one zero eigenvalue since $\int R(j,k)(\delta(k-\pi/2) - \delta(k+\pi/2))dk = 0$

Spectrum of decay rates of the current-correlation function

The $b = 0$ scattering operator has zero but also infinitely large (negative) eigenvalues. Which dominate the current-correlation function? Within this framework all currents relax multi-exponentially:

$$\hat{j} = \sum_k v_k a_k^\dagger a_k \quad \langle \hat{j}(t)\hat{j}(0)\rangle := \sum_e O(e)^2 \exp\{-et\}$$

"spectrum of decay rates"



- a substantial fraction of the current relaxes arbitrarily slow
- this graph appears to be idependent of the "coarse graining"
- one may calculate a time dependent diffusion coefficient from

$$D(t) = \int_0^t \mathrm{d}\tau \langle \hat{j}(\tau)\hat{j}(0)\rangle$$

Proceeding this way yields for large times $D(t) \propto \ln(t)$

Summary

We investigated the diffusion coefficient of a 1-d (spinless) fermionic system with a nearest neighbor hopping of $J = 1$, a next-nearest neighbor hopping of b, and a nearest neighbor interaction of Δ.

We find in the limit of small Δ for the inverse diffusion coefficient ($\propto$ resistivity):

$$D^{-1} \approx 0.25\ \sqrt{b}\ \Delta^2 + ?\Delta^4 + \cdots$$

Within this approach transport coefficients scale at least like Δ^{-4} for no next-nearest neighbor hopping.

All of this has been done for infinte temperature, $T = \infty$, however, the overall behavior is expected to remain unaltered at finite temperatures

Thanks !

If you are interested in Refs. just ask me.

We thank you, the audience, for your interest and patience!