Typicality Approach to Relaxation in Finite Quantum Systems

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Outline

- Why typicality?
- Do small “non-Markovian” quantum systems approach equilibrium?
- The typicality scenario
- Typicality in QM
- Hilbertspace average and variance of observables
- Exponential, rate-type relaxation of closed quantum systems
- Exponential relaxation of an energy exchange model
- Dynamical typicality
- The end

the naive view on relaxation / 2nd law of thermodynamcis:

QM: $\hat{\rho}_0 = |\psi\rangle\langle\psi|$ evolves into $\hat{\rho}_{eq} = \frac{1}{Z} e^{-\frac{\hat{H}}{kT}}$, $\hat{\rho}_{eq} = \frac{1}{Z}\hat{\delta}(\hat{H} - E)$

CM: $\rho_0 = \delta(x - x_0)\delta(p - p_0)$ evolves into $\rho_{eq} = \frac{1}{Z} e^{-\frac{H(x,p)}{kT}}$, $\rho_{eq} = \frac{1}{Z}\delta(H - E)$

issues, problems:

invariance of Von Neuman-entropy/phase-space volume, ergodicity, mixing, etc. framework of open systems: structure of environment (large, broad band, stationary oscillator bath) adequate weak coupling, applicability of projection techniques, initial states, etc.

Typicality:

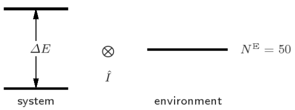
$\hat{\rho}_0 = |\psi\rangle\langle\psi|$ does not evolve into $\hat{\rho}_{eq} = \frac{1}{Z} e^{-\frac{\hat{H}}{kT}}$, $\hat{\rho}_{eq} = \frac{1}{Z}\hat{\delta}(\hat{H} - E)$

but $\langle\psi|\hat{A}(t)|\psi\rangle \Rightarrow \quad \approx \text{Tr}\{\hat{\rho}_{eq}\hat{A}\}$

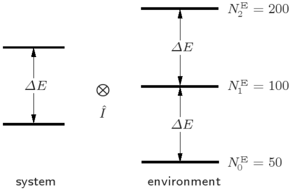
for very many (all?) $\hat{A}$

degenerate energy levels, weak interactions modelled by random matrices

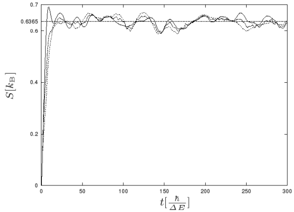
"microcanonical model"



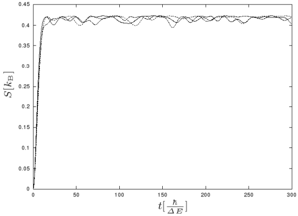
"canonical model"



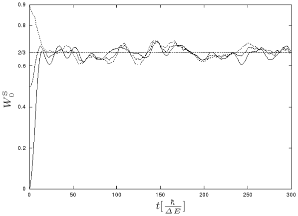
entropy evolution



entropy evolution

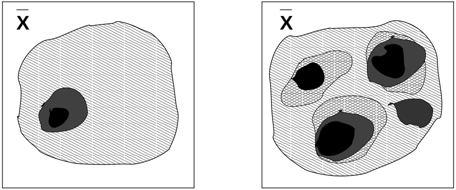


groundstate prob. evolution



- almost impossible to see recurrences
- typical evolution, almost independent of pure initial state
- all increase of entropy implies increase of entanglement

The typicality scenario



(original idea probably due to Ehrenfest and Boltzmann)

$\overline{x}$: microstates in state space, AR: "accessible region" due to constants of motion, etc. , $f(\overline{x})$: considered quantity

- Average: $E_{AR}(f) = \int_{AR} f(\overline{x})\, dV_{\overline{x}}$
- Variance: $V_{AR}(f) = E_{AR}[f^2] - E_{AR}[f]^2$

Typicality: $V_{AR}^{\frac{1}{2}}[f] \leq f_{\max} - f_{\min}$

$\Rightarrow$ relative frequency of stats featuring $f(\overline{x}) \approx E_{AR}(f)$ is high

connection to dynamics possible if $\dot{\overline{x}} = \overline{G}(\overline{x}) \qquad div_{\overline{x}}\overline{G} = 0$

invariance of state space volume, no ergodicity, mixing, etc.

parametrization of Hilbertspace

$$|\psi\rangle = \sum_n \psi_n\,|n\rangle = \sum_n (\eta_n + i\,\xi_n)|n\rangle$$

$\overline{x} = \{\eta_n, \xi_n\}$: η_n, ξ_n : real cartesian coordianates

dynamics: Schrödinger equation, $\dot{\overline{x}} = \overline{H}(\overline{x})$ $\quad div_{\overline{x}}\overline{H} = 0$

accessible region: $\hat{\Pi}_\alpha$: projective constants of motion (invariant subspaces), e.g., spanned by energy eigenstates, spanned by states featuring equal particle number, etc.

$$\hat{\Pi}_\alpha^2 = \hat{\Pi}_\alpha \quad , \quad \left[\hat{H}, \hat{\Pi}_\alpha\right] = 0 \quad , \quad N_\alpha = \mathrm{Tr}\{\hat{\Pi}_\alpha\} \quad , \quad \mathrm{Tr}\{\hat{\Pi}_\alpha, \hat{\Pi}_\beta\} = N_\alpha \delta_{\beta\alpha}$$

AR : $\{\langle\psi|\hat{\Pi}_\alpha|\psi\rangle = W_\alpha\}$ occupation probabilities of subspaces conserved

considered quantity : $f(\overline{x}) = \langle\psi|\hat{A}|\psi\rangle$

Hilbertspace average of observables:

$$E_{AR}[\langle\psi|\hat{A}|\psi\rangle] = \mathrm{Tr}\{\hat{A}\hat{\Omega}\} \quad , \quad \hat{\Omega} = \sum_{\alpha} \frac{W_\alpha}{N_\alpha}\hat{\Pi}_\alpha$$

Boltzmann state featuring "equal a priori probabilities"

Hilbertspace variance of observables: $(\hat{A}_{\alpha\beta} \equiv \hat{\Pi}_\alpha \hat{A} \hat{\Pi}_\beta)$

$$V_{AR}[\langle\psi|\hat{A}|\psi\rangle] = \sum_{\alpha\beta} \frac{W_\alpha W_\beta}{N_\alpha(N_\beta + \delta_{\alpha\beta})}\left(\mathrm{Tr}\{\hat{A}_{\alpha\beta}\hat{A}^\dagger_{\alpha\beta}\} - \delta_{\alpha\beta}\frac{\mathrm{Tr}\{\hat{A}_{\alpha\alpha}\}^2}{N_\alpha}\right)$$

consider, e.g.: ($\Delta^2_S(\hat{A})$: spectral variance)

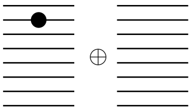
$$\langle\psi|\hat{\Pi}_\alpha|\psi\rangle = 1 \quad \Rightarrow \quad V_{AR}[\langle\psi|\hat{A}|\psi\rangle] = \frac{1}{N_\alpha + 1}\Delta^2_S(\hat{A})$$

typicality requires high dimensional Hilbert spaces, bounded spectra

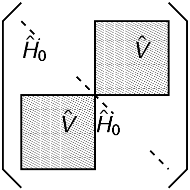
Exponential, rate-type relaxation of closed quantum systems

broad bands, weak hoppings modelled by random matrices

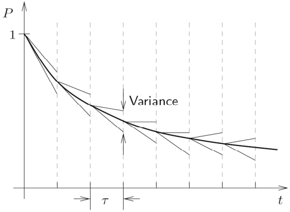
"two-site hopping model"



position operator: $\hat{A}$

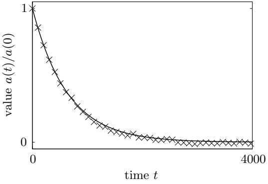


"iterative guessing":



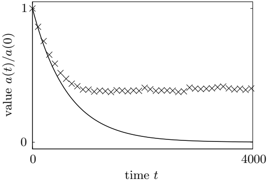
mean position vs. time

V_{ij} random



mean position vs. time

V_{ij} sparse



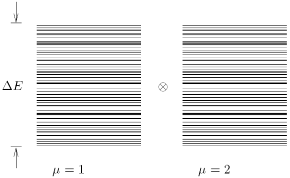
$$\mathrm{AR}: \{\langle\psi|\hat{A}(t)|\psi\rangle = A(t)\}$$

$$E_{AR}[\langle\psi|\hat{A}(t+\tau)|\psi\rangle] \approx (1 - R\tau)\langle\psi|\hat{A}(t)|\psi\rangle$$

$$V_{AR}[\langle\psi|\hat{A}(t+\tau)|\psi\rangle] \propto \frac{\tau}{n}$$

Each iterative guessing scheme corresponds to a concrete projective (TCL) approach

"two-subunit product model"



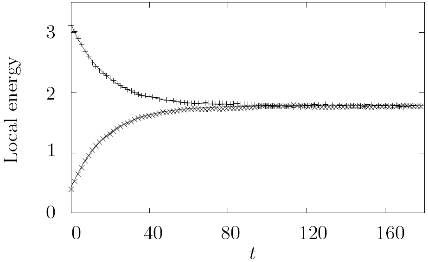
broad bands, weak interaction modelled by random matrix

$$\Delta E = 7$$

$$n = 60$$

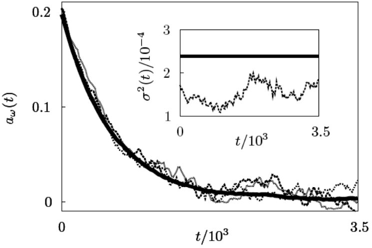
$$T_1 = 40, \quad T_2 = 1$$

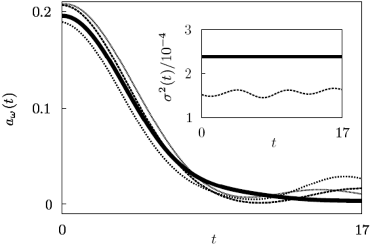
local energy evolution



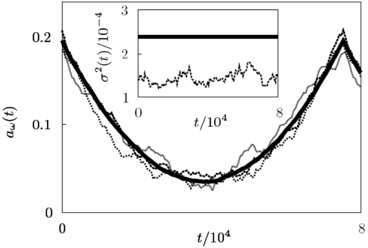
There is exponential relaxation towards equal distribution of local energies

Dynamical typicality





absolutly arbitrary models



$$\text{AR} : \{\langle\psi|\hat{A}|\psi\rangle = a(0)\}$$

$$V_{AR}[\langle\psi|\hat{A}(t)|\psi\rangle] \leq \frac{1}{n}$$

(mathematically) Almost all expectation value evolutions starting from a common initial value stay closely together

The end

Massively incomplete list of people who have worked or are working on this: Schrödinger, Von Neumann, Bocchieri + Loigner, Lubkin, Tasaki, Jensen + Shankar, LLoyd, Goldstein + Lebowitz + Tumulka + Zhangi, Reimann, Popescu + Short + Winter, Fine, Eisert, people from the audience, etc.

Pretty much everything in this talk has been published, in you are interested in references ask me, write me an email ...

Many Thanks to Christian Bartsch, Mathias Michel and Heinz Peter Breuer!

...and You, the audience for having listened carefully!